LP and simplex algorithm

optimization US = PROGRAMMING UK

What is LP

Linear Programming

Generally speaking, all problems with linear objective and linear equalities\inequalities constraints could be considered as Linear Programming. However, there are some widely accepted formulations.

Standard form

This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

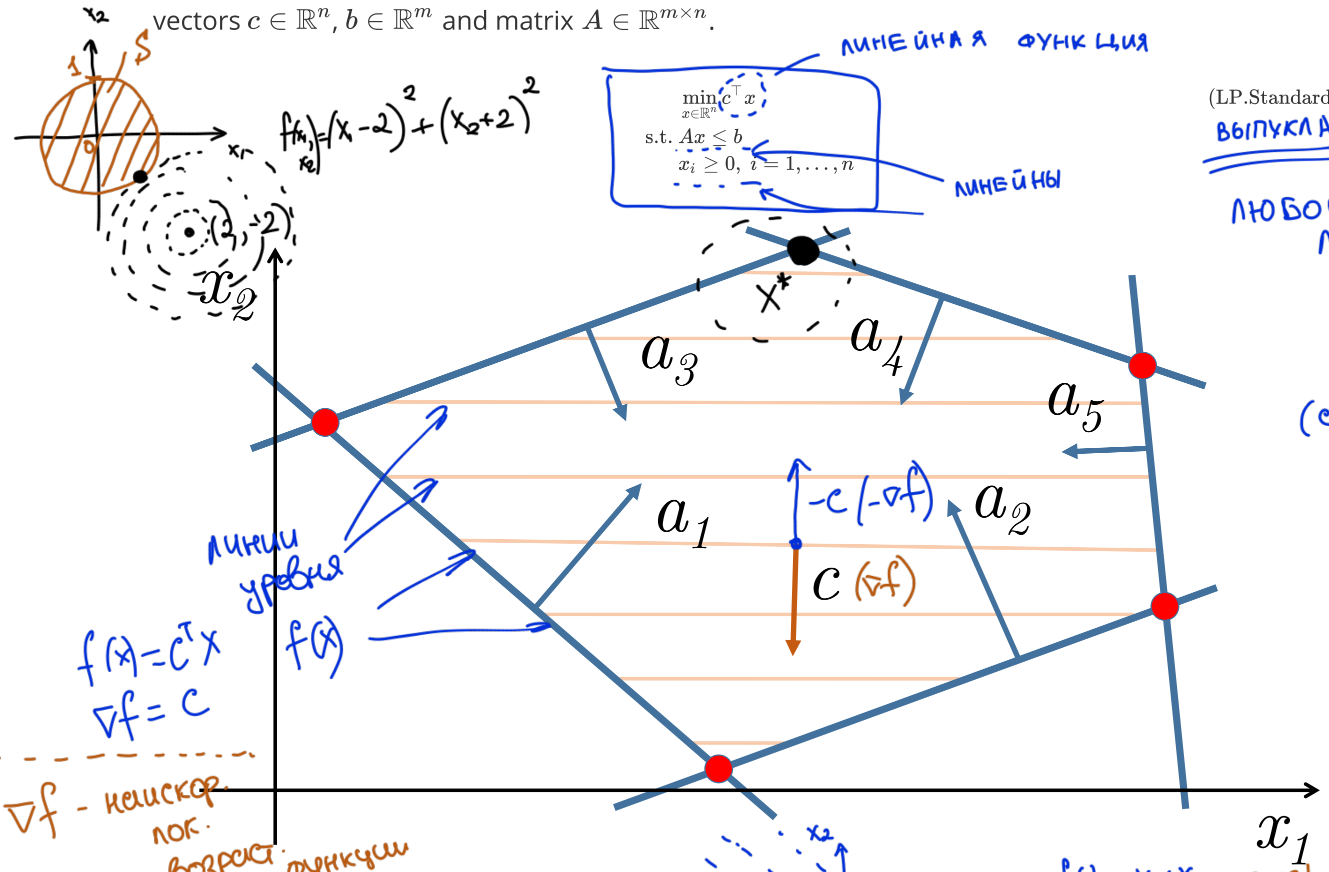
линейная функция

$$
\begin{split}
&\min_{x \in \mathbb{R}^n} c^{\top}x \\
\text{s.t. } & Ax \leq b\\
& x_i \geq 0, \; i = 1,\dots, n
\end{split}
\tag{LP.Standard}
$$

линейны

выпуклая

любой локальный минимум = глобальный (с т. зрения значения функции)



$f(x) = c^T x$

$\nabla f = c$

∇f - наискор. лок. возраст. функции

$-\nabla f$ - убывание

Canonical form

Ник. $x_N^* = \dots$

$f(x_N^*) \simeq 10$

Алекс $x_A^* = \dots$

$f(x_A^*) \simeq 10$

$\|x_N^* - x_A^*\|$ может быть очень большой

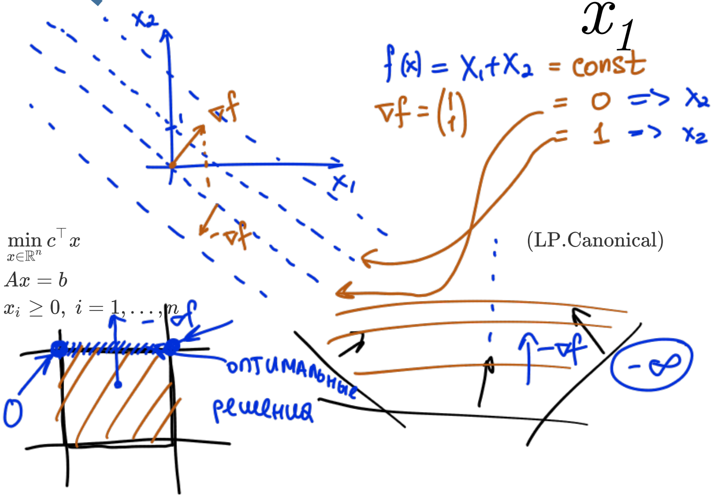
$f(x) = x_1 + x_2 = const$

$\nabla f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= 0 \Rightarrow x_2 = -x_1$

$= 1 \Rightarrow x_2 = 1 - x_1$

$$
\begin{split}
&\min_{x \in \mathbb{R}^n} c^{\top}x \\
\text{s.t. } & Ax = b\\
& x_i \geq 0, \; i = 1,\dots, n
\end{split}
\tag{LP.Canonical}
$$



Real world problems

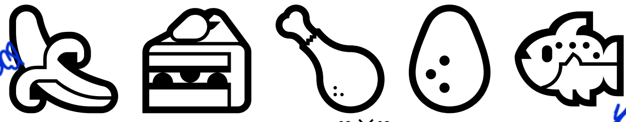
$|f(x_N^*) - f(x_A^*)| \sim 0$

Diet problem

Imagine, that you have to construct a diet plan from some set of products: 🍌🍰🍗🥚🐟. Each of the products has its own vector of nutrients. Thus, all the food information could be processed through the matrix W. Let also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^p} c^\top x \\
\text{s.t. } & Wx \geq r \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
$$

P штук разных продуктов



данные

Requirements

n нутриентов, о которых мы беспокоимся

$W \in \mathbb{R}^{n \times p}$, $\quad r \in \mathbb{R}^n$

$$
\begin{array}{r|c}
\text{Proteins} & \begin{pmatrix} \cdot & 3 & \cdot & 20 & \cdot \\ \cdot & 25 & \cdot & 5 & \cdot \\ & 25 & & 15 & \\ \cdot & 400 & \cdot & 200 & \cdot \\ \cdot & 100 & \cdot & 50 & \cdot \end{pmatrix} \\
\text{Carbs} & \\
\text{Fats} & \\
\text{Calories} & \\
\text{Vitamin D} & \\
\text{Вкусность} & \quad 1000 \qquad\qquad 100
\end{array}
\qquad
\begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix}
$$

$K \cdot x_1 + K_2 x_2 + \dots + K_p \cdot x_p \geq r$

$\boxed{c \in \mathbb{R}^p}$ - cost per 100 g

cost

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^p} c^\top x \\
&x \geq 0 \\
&Wx \geq r_{min}
\end{aligned}
$$

x – кол-во продуктов, которое мы хотим купить

$x \in \mathbb{R}^p$

$c_1 x_1 + \dots + c_p x_p \leftarrow$ стоимость заказа

$c^T x$

$Wx \leq r_{max}$.

Radiation treatment

How to retrieve LP

Basic transformations

Inequality to equality by increasing the dimension of the problem by m.

$$
Ax \leq b \leftrightarrow \begin{cases} Ax + z = b \\ z \geq 0 \end{cases}
$$

unsigned variables to nonnegative variables.

$$
x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}
$$

Chebyshev approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_i |a_i^\top x - b_i|
$$

$$
\begin{aligned}
&\min_{t \in \mathbb{R}, x \in \mathbb{R}^n} t \\
\text{s.t. } & a_i^\top x - b_i \leq t, \; i = 1, \dots, n \\
& - a_i^\top x + b_i \leq t, \; i = 1, \dots, n
\end{aligned}
$$

l_1 approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^\top x - b_i|
$$

$$
\begin{aligned}
\min_{t \in \mathbb{R}^n, x \in \mathbb{R}^n} \quad & \mathbf{1}^\top t \\
\text{s.t. } & a_i^\top x - b_i \le t_i, \; i = 1, \dots, n \\
& -a_i^\top x + b_i \le t_i, \; i = 1, \dots, n
\end{aligned}
$$

Idea of simplex algorithm

- The Simplex Algorithm walks along the edges of the polytope, at every corner choosing the edge that decreases $c^\top x$ most
- This either terminates at a corner, or leads to an unconstrained edge ($-\infty$ optimum)

We will illustrate simplex algorithm for the simple inequality form of LP:

$$
\begin{aligned}
& \min_{x \in \mathbb{R}^n} c^\top x \\
& \text{s.t. } Ax \le b
\end{aligned}
\tag{LP.Inequality}
$$

Definition: a **basis** B is a subset of n (integer) numbers between 1 and m, so that $\text{rank} A_B = n$. Note, that we can associate submatrix A_B and corresponding right-hand side b_B with the basis B. Also, we can derive a point of intersection of all these hyperplanes from basis: $x_B = A_B^{-1} b_B$.

If $Ax_B \le b$, then basis B is **feasible**.

$Ax \leq b$

A basis B is optimal if x_B is an optimum of the LP.Inequality.

0. начинаем с какой-нибудь вершины ✓

1. Проверяем оптимальность. ✓

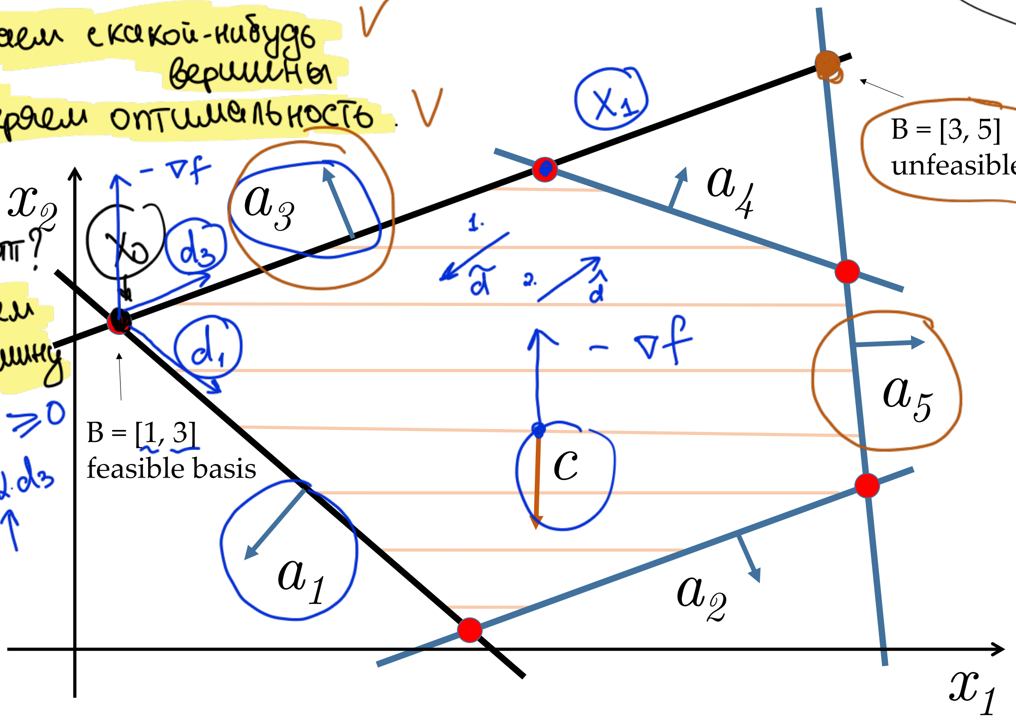
опт? END

неопт?

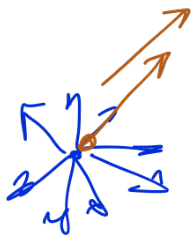
2. Меняем вершину

$d_3^T(-c) \geq 0$

$X_1 = X_0 + \alpha d_3$



Проекц. $(-\nabla f)$ на пл-ть



Since we have a basis, we can decompose our objective vector c in this basis and find the scalar coefficients λ_B:

$\begin{bmatrix} a_1 \\ a_3 \end{bmatrix}_{n \times n}$

$$
\lambda^\top_B A_B = c^\top \leftrightarrow \lambda^\top_B = c^\top A_B^{-1}
$$

1. $\lambda \leq 0$

$c = \sim a_1 + \dots \sim a_3$

Симплекс алгоритм

Main lemma

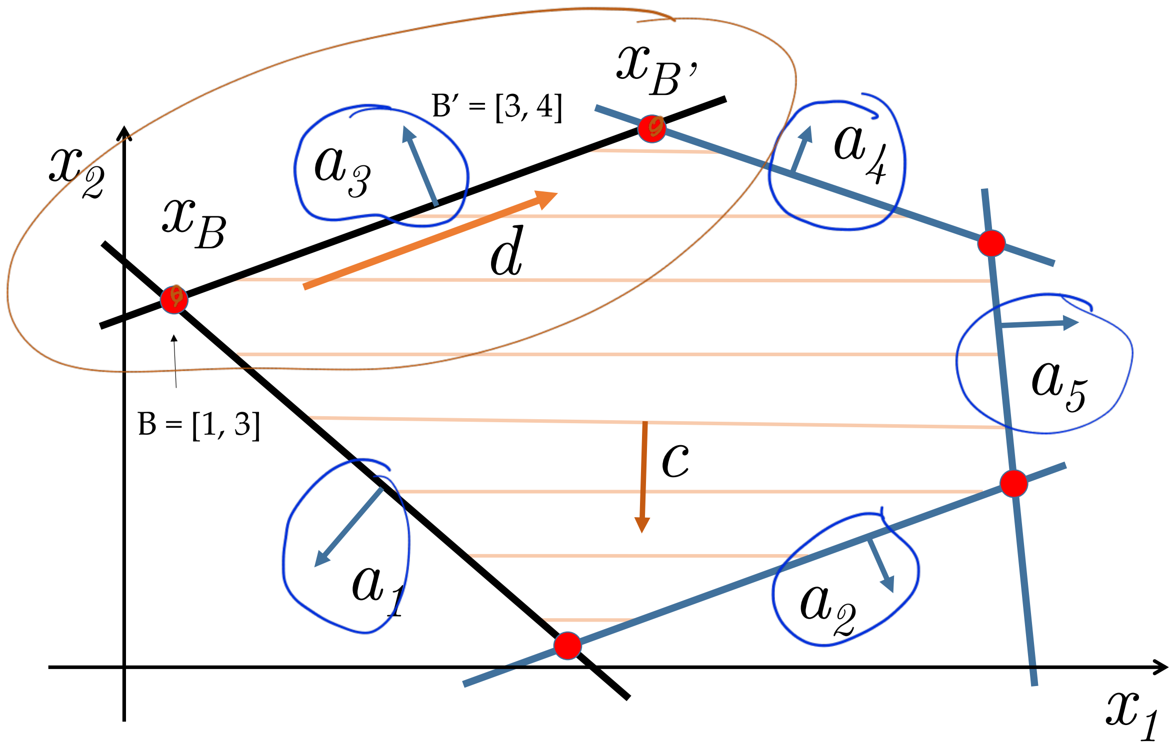
If all components of λ_B are non-positive and B is feasible, then B is optimal.

Proof:

$$
\begin{aligned}
\exists x^* : Ax^* &\leq b, c^\top x^* < c^\top x_B \\
A_B x^* &\leq b_B \\
\lambda_B^\top A_B x^* &\geq \lambda_B^\top b_B \\
c^\top x^* &\geq \lambda_B^\top A_B x_B \\
c^\top x^* &\geq c^\top x_B
\end{aligned}
$$

Changing basis

Suppose, some of the coefficients of λ_B are positive. Then we need to go through the edge of the polytope to the new vertex (i.e., switch the basis)



$$
x_{B'} = x_B + \mu d = A_{B'}^{-1} b_{B'}
$$

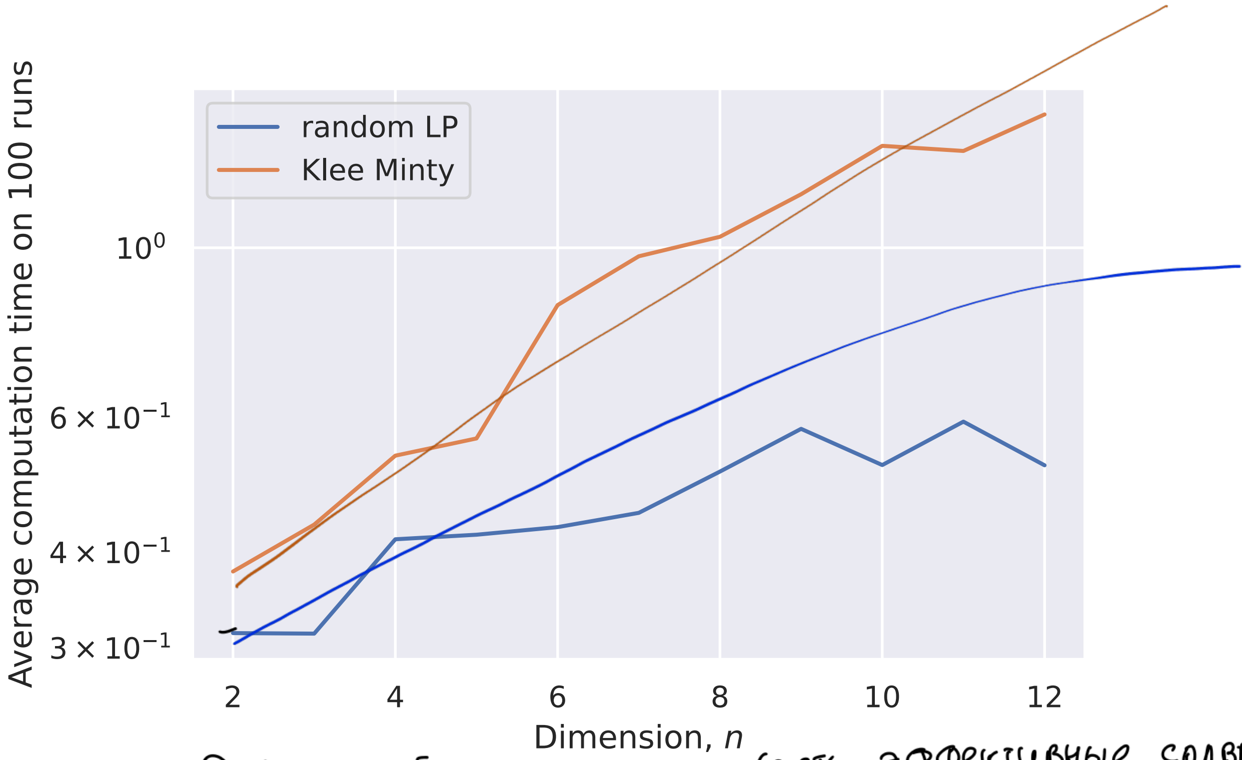
About convergence

Klee Minty example

в худшем случае алгоритм работает экспоненц. долго.

In the following problem simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$
\begin{aligned}
& \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + x_n \\
\text{s.t. } & x_1 \leq 5 \\
& 4x_1 + x_2 \leq 25 \\
& 8x_1 + 4x_2 + x_3 \leq 125 \\
& \dots \\
& 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \ldots + x_n \leq 5^n \qquad x \geq 0
\end{aligned}
$$



⊕ Довольно быстро решается (есть эффективные солверы)

- Отсутствует сильная выпуклость (НЕТ СХОДИМОСТИ ПО АРГУМЕНТУ)
- Симплекс метод прост, но в худшем случае работает экспоненциально долго.
- Есть много библиотек (scipy, pulp, pyomo)

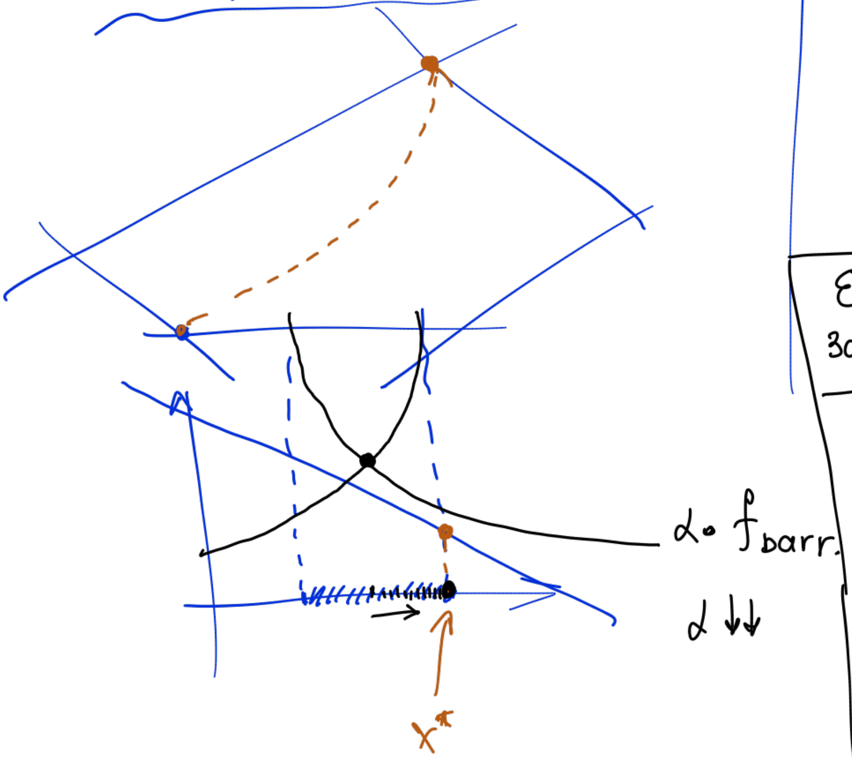
Code

[Open in Colab](#)

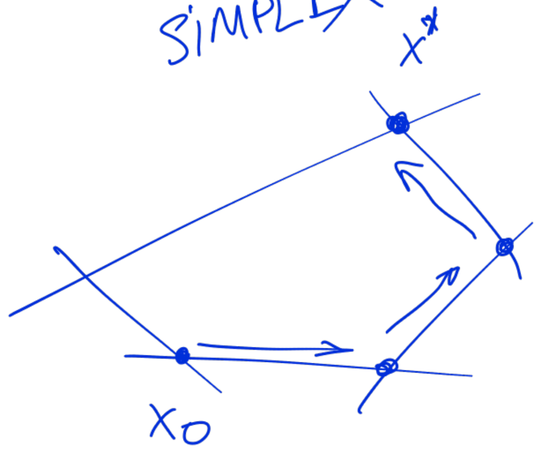
Materials

- [Linear Programming.](#) in V. Lempitsky optimization course.
- [Simplex method.](#) in V. Lempitsky optimization course.
- [Overview of different LP solvers](#)
- [TED talks watching optimization](#)

Методы внутренней точки

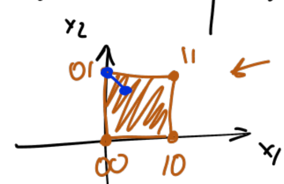


SIMPLEX



Если часть переменных - дискретные. Задача сильно усложняется.

N тумблеров $\left\{\begin{matrix}0\\1\end{matrix}\right\}$ 2^N вариантов



не выпуклое бюджетное мн-во

задачи

MIXED INTEGER PROGRAMMING