LP and simplex algorithm

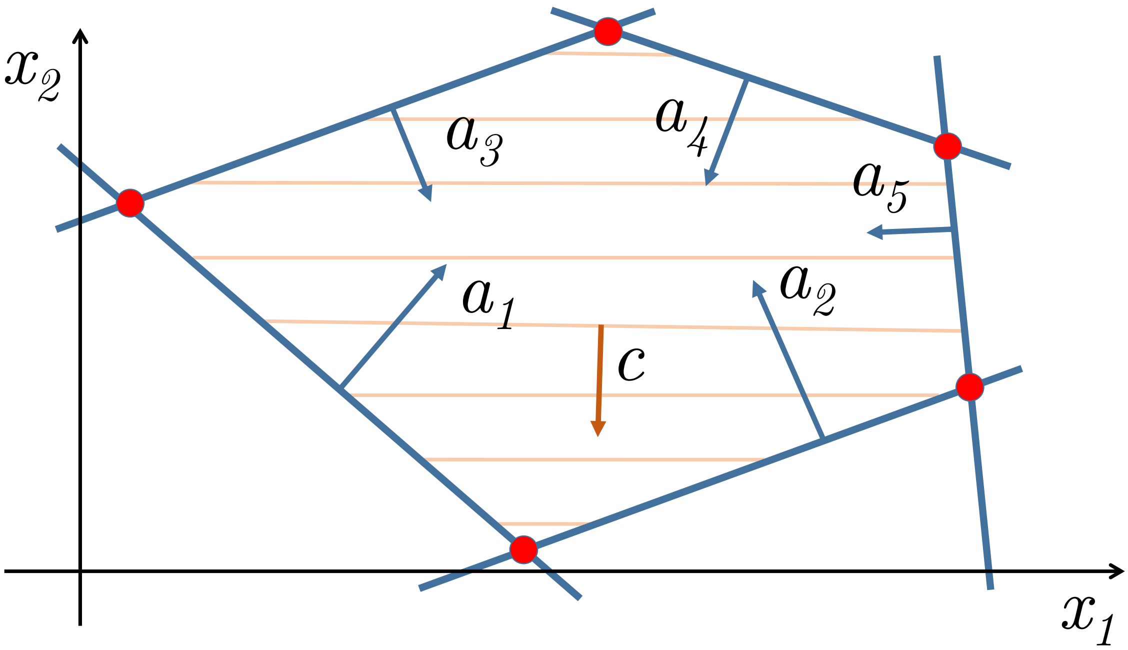
What is LP

Generally speaking, all problems with linear objective and linear equalities\inequalities constraints could be considered as Linear Programming. However, there are some widely accepted formulations.

Standard form

This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } &Ax \leq b \\
&x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Standard}
$$



Canonical form

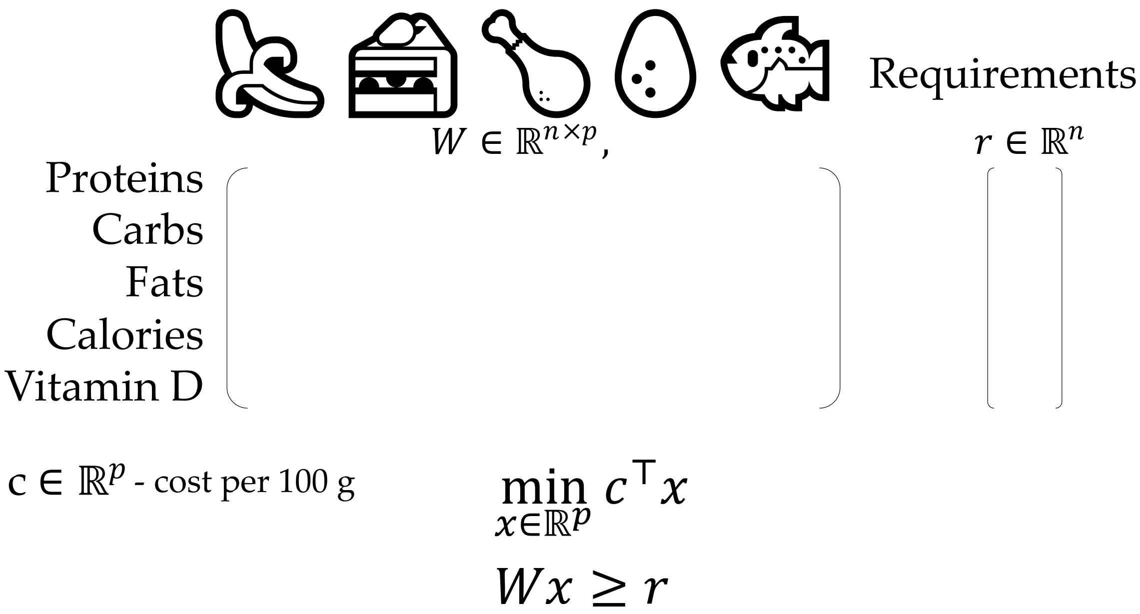
$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } &Ax = b \\
&x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Canonical}
$$

Real world problems

Diet problem

Imagine, that you have to construct a diet plan from some set of products: 🍌🍰🍗🥚🐟. Each of the products has its own vector of nutrients. Thus, all the food information could be processed through the matrix W. Let also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^p} c^{\top}x \\
\text{s.t. } & Wx \geq r \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
$$



Radiation treatment

How to retrieve LP

Basic transformations

Inequality to equality by increasing the dimension of the problem by m.

$$
Ax \leq b \leftrightarrow \begin{cases} Ax + z = b \\ z \geq 0 \end{cases}
$$

unsigned variables to nonnegative variables.

$$
x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}
$$

Chebyshev approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_i |a_i^\top x - b_i|
$$

$$
\begin{aligned}
&\min_{t \in \mathbb{R}, x \in \mathbb{R}^n} t \\
\text{s.t. } & a_i^\top x - b_i \leq t, \; i = 1, \dots, n \\
& - a_i^\top x + b_i \leq t, \; i = 1, \dots, n
\end{aligned}
$$

l_1 approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^\top x - b_i|
$$

$$
\begin{split}
&\min_{t \in \mathbb{R}^n, x \in \mathbb{R}^n} \mathbf{1}^\top t \\
\text{s.t. } & a_i^\top x - b_i \leq t_i, \; i = 1, \dots, n \\
& -a_i^\top x + b_i \leq t_i, \; i = 1, \dots, n
\end{split}
$$

Idea of simplex algorithm

- The Simplex Algorithm walks along the edges of the polytope, at every corner choosing the edge that decreases $c^\top x$ most
- This either terminates at a corner, or leads to an unconstrained edge ($-\infty$ optimum)

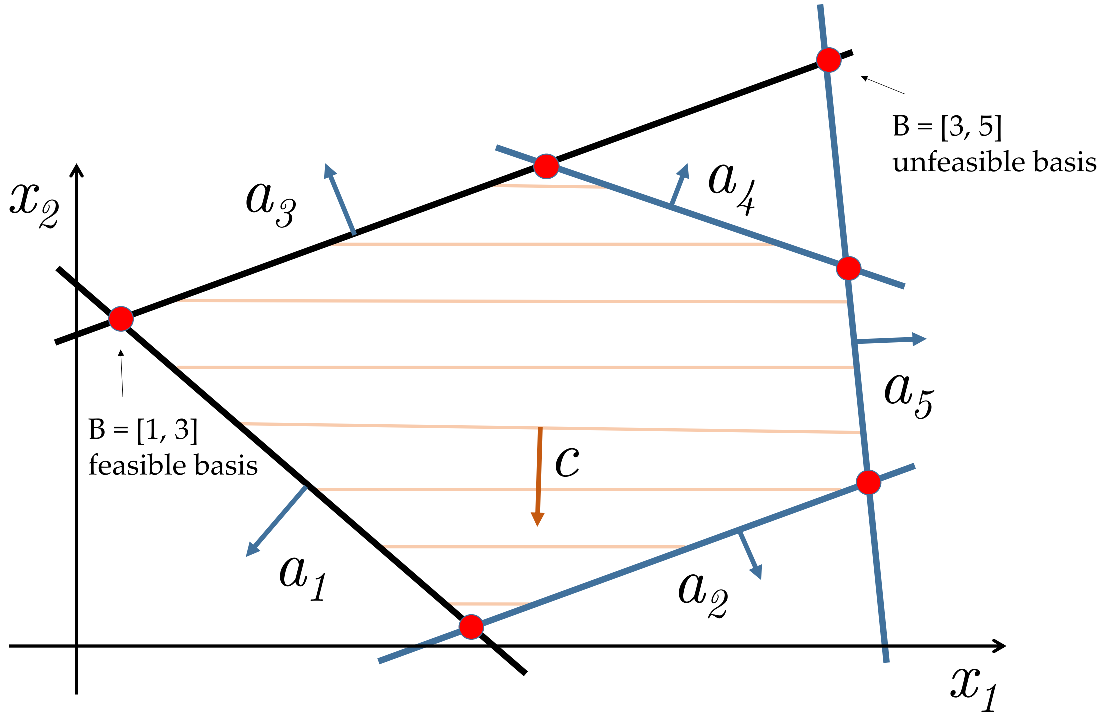
We will illustrate simplex algorithm for the simple inequality form of LP:

$$
\begin{split}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b
\end{split}
\tag{LP.Inequality}
$$

Definition: a **basis** B is a subset of n (integer) numbers between 1 and m, so that $\text{rank} A_B = n$. Note, that we can associate submatrix A_B and corresponding right-hand side b_B with the basis B. Also, we can derive a point of intersection of all these hyperplanes from basis: $x_B = A_B^{-1} b_B$.

If $Ax_B \leq b$, then basis B is **feasible**.

A basis B is optimal if x_B is an optimum of the LP.Inequality.



Since we have a basis, we can decompose our objective vector c in this basis and find the scalar coefficients λ_B:

$$
\lambda_B^\top A_B = c^\top \leftrightarrow \lambda_B^\top = c^\top A_B^{-1}
$$

Main lemma

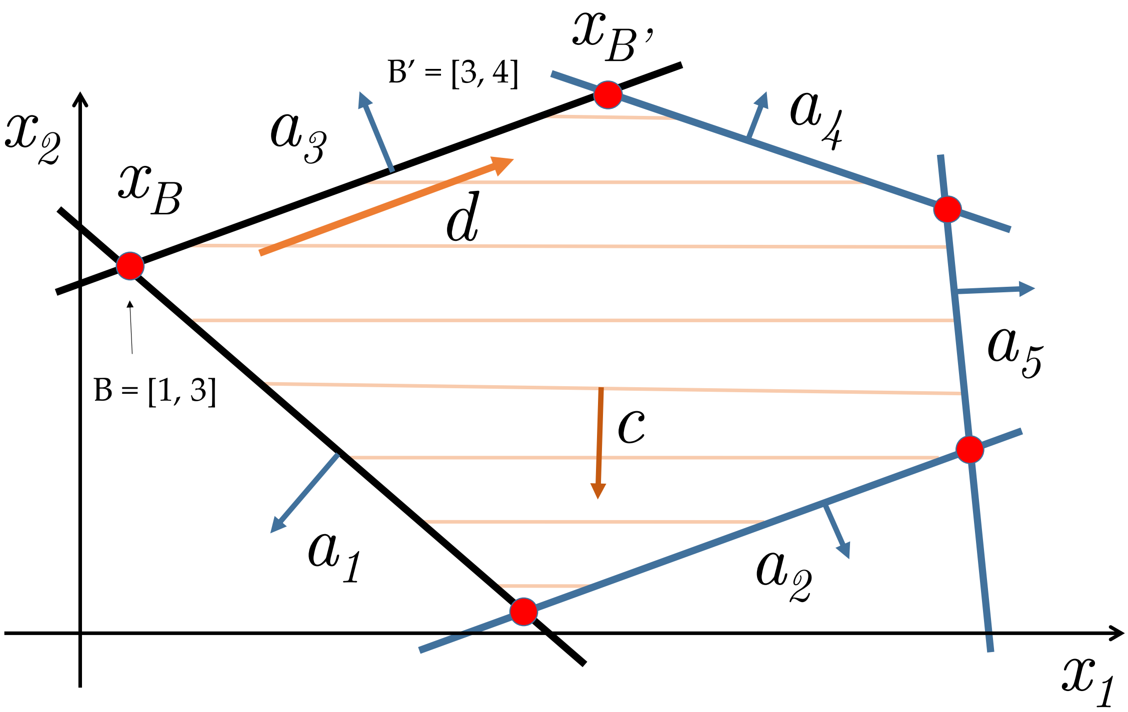
If all components of λ_B are non-positive and B is feasible, then B is optimal.

Proof:

$$
\begin{aligned}
\exists x^* : Ax^* \leq b, c^\top x^* < c^\top x_B \\
A_B x^* \leq b_B \\
\lambda_B^\top A_B x^* \geq \lambda_B^\top b_B \\
c^\top x^* \geq \lambda_B^\top A_B x_B \\
c^\top x^* \geq c^\top x_B
\end{aligned}
$$

Changing basis

Suppose, some of the coefficients of λ_B are positive. Then we need to go through the edge of the polytope to the new vertex (i.e., switch the basis)



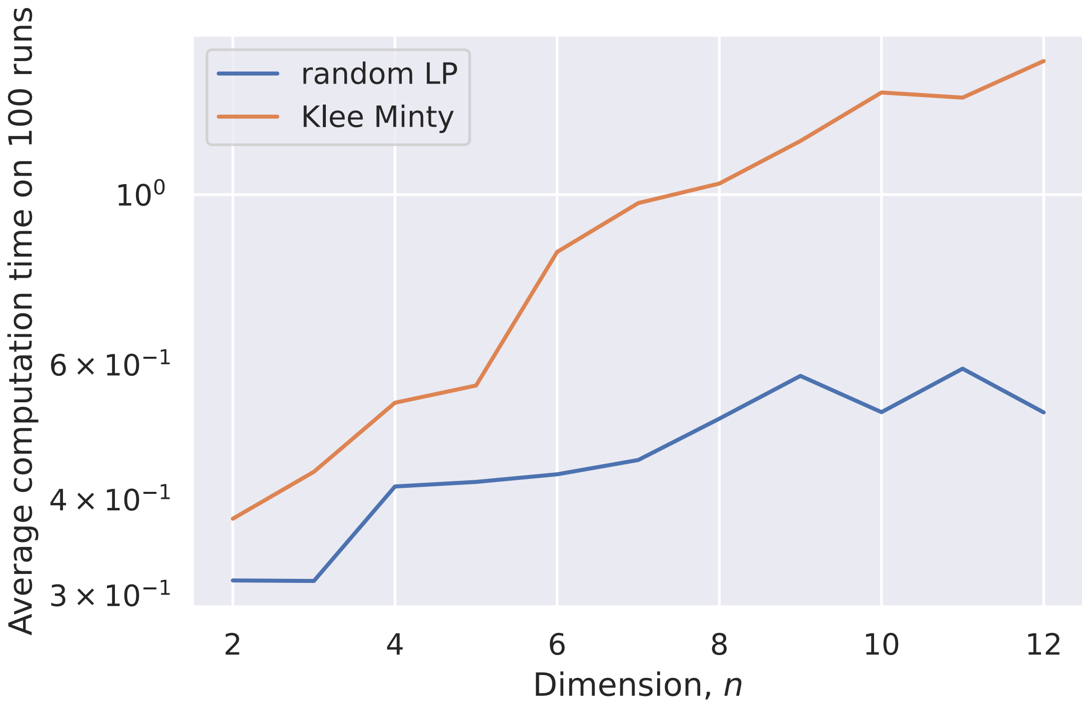
$$
x_{B'} = x_B + \mu d = A_{B'}^{-1} b_{B'}
$$

About convergence

Klee Minty example

In the following problem simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$
\begin{aligned}
& \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + x_n \\
\text{s.t. } & x_1 \leq 5 \\
& 4x_1 + x_2 \leq 25 \\
& 8x_1 + 4x_2 + x_3 \leq 125 \\
& \ldots \\
& 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \ldots + x_n \leq 5^n \qquad x \geq 0
\end{aligned}
$$



Code

[Open in Colab]

Materials

- Linear Programming. in V. Lempitsky optimization course.
- Simplex method. in V. Lempitsky optimization course.
- Overview of different LP solvers
- TED talks watching optimization